Contributions in Modelling and Control of Traffic Flow

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Outline

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Motivation

- Congestion often appears without any accident:
  small perturbations amplify into a stop-and-go wave[2]
- This is a phantom jam: pure consequence of density and reaction delays
- Traffic behaves like a compressible fluid: it self-organizes into shocks

$\Rightarrow$ Societal, economic and environmental cost: hours lost daily, emissions, accident risk

[2] Sugiyama, Fukui, Kikuchi, Hasebe, Tanaka, Tomoeda, and Tsuchiya 2008.

General setting

- Evolution of a scalar quantity (mass density of a medium) in one space dimension
- Distribution is described by a density

$$\rho \in L^1_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}), \tag{1}$$

- For an interval $[x_0, x_1]$, total amount of matter is

$$\int_{x_0}^{x_1} \rho(t, x)\, dx, \tag{2}$$

- Flux function measuring what crosses boundary of $[x_0, x_1]$

$$F \in L^1_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}), \tag{3}$$

- Conservation principle yields for $t_1 < t_2$

$$\int_{x_0}^{x_1} \rho(t_2, x)\, dx = \int_{x_0}^{x_1} \rho(t_1, x)\, dx + \int_{t_1}^{t_2} F(s, x_0)\, ds - \int_{t_1}^{t_2} F(s, x_1)\, ds, \tag{4}$$

General setting

- If ρ and F are smooth enough, then integral balance is equivalent [3] to local PDE

$$\partial_t \rho(t,x) + \partial_x F(t,x) = 0, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}_+, \tag{5}$$

- To obtain deterministic PDE, specify a constitutive law relating flux and density
  - *Burgers' equation*

$$F(t,x) = \frac{\rho(t,x)^2}{2} \quad \Longrightarrow \quad \partial_t \rho + \partial_x \left(\frac{\rho^2}{2} \right) = 0, \tag{6}$$

  - *Heat equation*

$$F(t,x) = -\kappa \, \partial_x \rho(t,x) \quad \Longrightarrow \quad \partial_t \rho = \kappa \, \partial_{xx} \rho, \tag{7}$$

  - *Lighhwill-Whitham-Richards equation*

$$F(t,x) = v_{\max} \, \rho(t,x) \left(1 - \frac{\rho(t,x)}{\rho_{\max}} \right), \tag{8}$$

- Focus on hyperbolic conservation laws

$$F(t,x) := f(x, \rho(t,x)), \quad x \in \mathbb{R}, \quad t \in \mathbb{R}_+, \tag{9}$$

[3] it follows from fundamental theorem of calculus applied to integral balance

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Macroscopic description of traffic

- Nonlinear [4] scalar conservation law in one space dimension
- Unknown variable $\rho(t,x)$ stands for traffic density
- Time evolution of mass contained in $[a,b]$ at $t \in \mathbb{R}^+$ is only affected by flux at edges

$$\frac{d}{dt}\int_a^b \rho(t,x)dx = f(\rho(t,a)) - f(\rho(t,b)), \quad t \in \mathbb{R}_+, \tag{10}$$

- Continuity [5] equation for ρ

$$\partial_t \rho + \partial_x f(\rho) = 0, \quad \mathbb{R}_+ \times \mathbb{R}, \tag{11}$$

- Vehicles instantaneously adjust to an equilibrium velocity[6] dictated by local density

$$v(t,x) := v(\rho(t,x)), \quad x \in \mathbb{R}, \quad t \in \mathbb{R}_+, \tag{12}$$

- Constitutive law for the Eulerian velocity of medium

$$f(\rho) = \rho v(\rho), \tag{13}$$

[4] nonlinearity has made mathematical theory a challenging topic
[5] medium is assumed to be indefinitely divisible without changing its physical nature
[6] traffic is treated as a compressible fluid: no acceleration effects, no reaction delay

LWR model

$$\begin{cases} \partial_t \rho(t,x) + \partial_x f(\rho)(t,x) = 0, & x \in \mathbb{R}, \quad t \in \mathbb{R}^+, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}, \end{cases} \tag{LWR}$$

- Reasonnable assumption: number of cars is conserved
  $\Rightarrow$ no source or sink on a single road segment
- v is typically required to be differentiable for well-posedness
  $\Rightarrow$ a stronger requirement than in microscopic (follow-the-leader) models
- **Main limitation**: model uses a local equilibrium closure
  $\Rightarrow$ cannot describe capacity drop, hysteresis or spontaneous stop-and-go waves

Why smooth data stops being smooth

- Along a characteristic curve $x(t)$ solving

$$\frac{dx}{dt}(t) = f'(\rho(t, x(t))), \tag{14}$$

(LWR) says ρ is constant

- Characteristics are straight lines

$$x(t) = x_0 + f'(\rho_0(x_0))t, \tag{15}$$

with slope fixed by initial datum at x_0

- Characteristics starting at $x_0 < x_0'$ with $\rho_0(x_0) < \rho_0(x_0')$ have [7] slopes

$$f'(\rho_0(x_0)) > f'(\rho_0(x_0')), \tag{16}$$

$\Rightarrow$ characteristics converge: they impinge on the shock

- Characteristics cross in finite time

$$t^\star = \min_{x_0 < x_0'} \frac{x_0' - x_0}{f'(\rho_0(x_0)) - f'(\rho_0(x_0'))}, \tag{17}$$

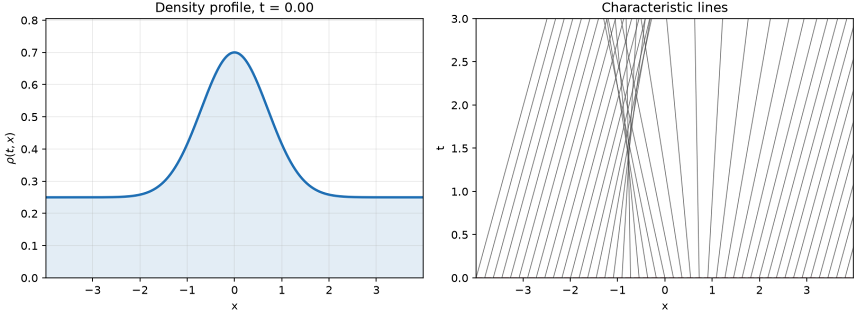
$\Rightarrow$ even smooth (C^1) initial data produces a discontinuous solution at $t = t^\star$

$\Rightarrow$ classical solutions cannot be extended [8] past $t^\star$

[7] for a concave f

[8] this is generic for nonlinear conservation laws

Shock formation: profile and characteristics



Left: density profile (Godunov scheme). Right: characteristic lines $x = x_0 + f'(\rho_0(x_0))\,t$.

Weak formulation

- To go past $t^\star$, we must accept less regular (possibly discontinuous) solutions ρ
- ρ is a "weak" solution to (LWR) if [9] for all $x_0 \leq x_1$, $t_0 < t_1$

$$\int_{x_0}^{x_1} \rho(t_1, x)\, dx + \int_{t_0}^{t_1} f(\rho(t, x_1))\, dt - \int_{x_0}^{x_1} \rho(t_0, x)\, dx - \int_{t_0}^{t_1} f(\rho(t, x_0))\, dt = 0, \qquad (18)$$

$\Rightarrow$ formulation makes sense for merely L^∞ solutions: no differentiability required

- A bounded function $\rho \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$ is a weak solution of (LWR) if

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} (\rho\, \partial_t \varphi + f(\rho)\, \partial_x \varphi)\, dx\, dt + \int_{\mathbb{R}} \rho_0(x)\varphi(0, x)\, dx = 0, \qquad (19)$$

for every test function $\varphi \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R})$

[9] the integral of the differential form $\rho\, dx - f(\rho)\, dt$ around the rectangle with corners $(x_0, t_0), (x_1, t_0), (x_1, t_1), (x_0, t_1)$ vanishes

Riemann problem

- If ρ solves (LWR), so does $\rho(\cdot, \cdot - a)$ for any a
  $\Rightarrow$ translation invariance: center the jump at $x_c = 0$ w.l.o.g
- If ρ solves (LWR), so does $\rho(\alpha\cdot, \alpha\cdot)$ for $\alpha > 0$
  $\Rightarrow$ scaling invariance: no intrinsic length or time scale
- Cauchy problem for (LWR) with simplest possible data: a single jump

$$\rho_0(x) = \begin{cases} \rho_l, & x < 0, \\ \rho_r, & x > 0, \end{cases} \tag{20}$$

- Search [10] for a self-similar (depends only on x/t) solutions
  1. a single shock (i.e. a single jump moving at constant unknown speed)

$$\rho(t,x) = \begin{cases} \rho_l, & x < \gamma t, \\ \rho_r, & x > \gamma t, \end{cases} \tag{21}$$

  2. a single rarefaction fan (i.e. continuous solutions)

$$\rho(t,x) = \begin{cases} \rho_l, & x < f'(\rho_l)\, t, \\ (f')^{-1}\left(\dfrac{x}{t}\right), & f'(\rho_l)\, t \le x \le f'(\rho_r)\, t, \\ \rho_r, & x > f'(\rho_r)\, t. \end{cases} \tag{22}$$

- General solutions (arbitrary ρ_0) are built [11] by gluing Riemann problems

[10] for a concave f, if f is convex then shocks become rarefactions and vice versa
[11] wave-front tracking, Godunov schemes

Admissibility condition

- Riemann datum $\rho_0 = \rho_l \, \mathbf{1}_{x<0} + \rho_r \, \mathbf{1}_{x>0}$
- A jump from ρ_l to ρ_r propagating [12] at speed s is admissible only if it satisfies

$$\gamma = s = \frac{f(\rho_l) - f(\rho_r)}{\rho_l - \rho_r}, \tag{23}$$

$\Rightarrow$ Rankine–Hugoniot condition[13] only way to solve Riemann problem globally in time

- Weak solutions are not unique
  $\Rightarrow$ several jumps can satisfy Rankine–Hugoniot condition without being physical

[12] with $x_0 = \min(\gamma t_0, \gamma t_1)$ and $x_1 = \max(\gamma t_0, \gamma t_1)$ it holds $\gamma\,(t_1 - t_0)(\rho_l - \rho_r) = (t_1 - t_0)\big(f(\rho_l) - f(\rho_r)\big)$

[13] this constant-speed jump is a necessary and sufficent condition (using Chasles' decomposition)

Admissibility condition

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$$\gamma = s = \frac{f(\rho_l) - f(\rho_r)}{\rho_l - \rho_r}, \tag{24}$$

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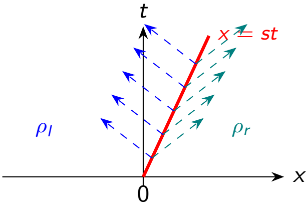
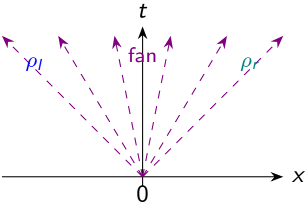


Candidate 1



Candidate 2

Same Riemann data ρ_l, ρ_r: both candidates solve LWR weakly and satisfy Rankine–Hugoniot — which one is physical?

- **Candidate 1 (shock)**

$$\rho(t,x) = \begin{cases} \rho_l, & x < s\,t, \\ \rho_r, & x > s\,t, \end{cases} \qquad s = \frac{f(\rho_l) - f(\rho_r)}{\rho_l - \rho_r} \tag{25}$$

- **Candidate 2 (rarefaction)**

$$\rho(t,x) = \begin{cases} \rho_l, & x < f'(\rho_l)\,t, \\ (f')^{-1}\left(\dfrac{x}{t}\right), & f'(\rho_l)\,t \le x \le f'(\rho_r)\,t, \\ \rho_r, & x > f'(\rho_r)\,t. \end{cases} \tag{26}$$

Selecting the physical solution

- Traffic jam correspond to shocks propagating backward along which density increases
- Entropy condition: characteristics must converge into the shock from both sides [16]

$$f'(\rho_l) > s > f'(\rho_r), \tag{27}$$

$\Rightarrow$ selects the physically meaningful solution

- A weak solution to (LWR) is entropy admissible[17] if

$$\begin{aligned}
&\int_{\mathbb{R}_+}\int_{\mathbb{R}} |\rho - k|\partial_t\phi + \mathrm{sign}(\rho - k)(f(\rho) - f(k))\partial_x\phi \, dx\, dt \\
&+ \int_{\mathbb{R}} |\rho_0 - k|\phi(0,x)dx \geq 0, \quad \forall k \in \mathbb{R},
\end{aligned} \tag{28}$$

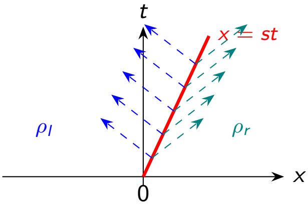
for every test function $\varphi \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R})$

- Equivalent construction: vanishing viscosity

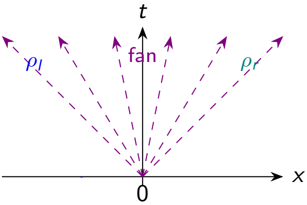
$$\partial_t\rho^\varepsilon + \partial_x f(\rho^\varepsilon) = \varepsilon\,\partial_{xx}\rho^\varepsilon, \quad \varepsilon \to 0, \tag{29}$$

$\Rightarrow$ recovers exactly the entropy solution[18]

[16] speed of admissible discontinuity is less than characteristic speed on the left and greater on the right one
[17] Kružkov 1970.
[18] Hopf 1950.



Shock: satisfies R–H
but characteristics *diverge*
$\Rightarrow$ violates entropy



Rarefaction: continuous,
characteristics *fan out*
$\Rightarrow$ the entropy solution

Same data $\rho_l > \rho_r$: both solve (LWR) weakly and satisfy Rankine–Hugoniot but only the rarefaction is entropy-admissible

Well-posedness of the model

- Entropy solutions enjoy a comparison principle

$$\rho_0 \le \tilde{\rho}_0 \quad \Longrightarrow \quad \rho(t) \le \tilde{\rho}(t), \quad \forall t \in \mathbb{R}_+, \tag{30}$$

  inherited directly from the underlying microscopic model
- Entropy solutions enjoy an L^1-contraction property

$$\|\rho(t) - \tilde{\rho}(t)\|_{L^1} \le \|\rho_0 - \tilde{\rho}_0\|_{L^1}, \quad \forall t \in \mathbb{R}_+, \tag{31}$$

  which gives uniqueness and stability
- **Take-away**: (LWR) model is now well-posed on a single road
  microscopic monotonicity $\Rightarrow$ macroscopic contraction $\Rightarrow$ entropy selection

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Conservation law with unilateral constraint

- Perturb single-road picture with a local capacity restriction at a point $x = 0$
  $\Rightarrow$ model a a toll gate, a lane reduction, a traffic light
- Consider (LWR) and impose a pointwise flux constraint

$$f\left(\rho(t,0)\right) \leq q(t), \quad t \in \mathbb{R}^+, \tag{32}$$

  where q is the maximal flow rate allowed through the gate [19]
- **Question**: does the entropy solution still make sense once we add constraint (32)?
- General solutions obtained by gluing Riemann problems
  $\Rightarrow$ understanding (32) reduces to understanding *one* Riemann problem at $x = 0$
- **Main difficulty**: the constraint acts *only* at $x = 0$
  $\Rightarrow$ may be incompatible [20] with the classical (unconstrained) entropy solution

[19] externally imposed capacity bound: at most $q(t)$ vehicles per unit time may cross $x = 0$
[20] need to modify the solver itself not just filter its output

LWR traffic model on a road

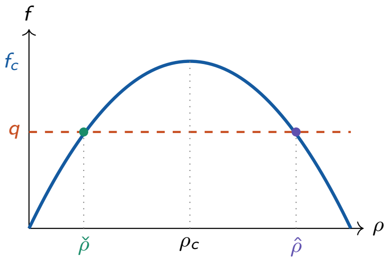
$$\partial_t \rho + \partial_x f(\rho) = 0, \quad \rho \in [0, \rho_{\max}] \tag{33}$$

- density ρ, flux $f(\rho)$ (*bell-shaped*)
- $f(0) = f(\rho_{\max}) = 0$, unique max $f_c = f(\rho_c)$
- free branch $\rho < \rho_c$ $(f' > 0)$,
  congested $\rho > \rho_c$ $(f' < 0)$

Pointwise constraint

$$f\big(\rho(t,0)\big) \;\leq\; q(t), \tag{34}$$

with $q \in \mathrm{BV}(\mathbb{R}^+; [0, f_c])$ given



The cap q selects two densities:
free trace $\check{\rho}$, congested trace $\hat{\rho}$,
$f(\check{\rho}) = f(\hat{\rho}) = q$

- Upstream: queue builds up
- Downstream: road stays fluid

Mathematical difficulty

Three obstructions

1. **Non-uniqueness**
   Weak solutions are not unique: an entropy criterion is needed
2. **The constraint forces a non-classical shock**
   When the flux would exceed q, the solver inserts a *stationary* jump

$$\hat{\rho} \,\Big|\, \check{\rho}, \quad \check{\rho} < \rho_c < \hat{\rho}, \quad f(\check{\rho}) = f(\hat{\rho}) = q, \tag{35}$$

   Since $f'(\hat{\rho}) < 0 < f'(\check{\rho})$ this violates Lax/Oleinik [a]
3. $\mathrm{TV}(\rho)$ **can jump in time**
   Each downward jump of q creates waves: a naive BV bound in ρ is unavailable

[a] it is admissible only because the constraint forces it

The trick[a]

[a] Colombo and Goatin 2007.

- Measure the solution not in ρ but in a *Temple coordinate*[a]
- Add a boundary term that "pays" for the non-classical shock

[a] Temple 1982.

Temple coordinate and Glimm functional

- Temple coordinate

$$\Psi(\rho) = \operatorname{sgn}(\rho - \rho_c)\big(f_c - f(\rho)\big), \tag{36}$$

strictly increasing with $\Psi(\rho_c) = 0$

- Key identity: the non-classical front satisfies

$$|\Psi(\hat\rho) - \Psi(\check\rho)| = 2(f_c - q), \tag{37}$$

- Glimm-type functional [21]

$$\Upsilon(\rho^n, q^n) = \underbrace{\sum_{\alpha} \big|\Psi(\rho^n_{\alpha+1}) - \Psi(\rho^n_{\alpha})\big|}_{\text{(A) spatial variation in } \Psi} + \underbrace{5 \sum_{t_\beta \geq 0} \big|q^n_{\beta+1} - q^n_{\beta}\big|}_{\text{(B) future variation of } q} + \underbrace{\gamma}_{\text{(C) state of the gate}}, \tag{38}$$

where

$$\gamma = \begin{cases} 0 & \text{if saturated (non-classical)}^{22}, \\ 4\big(f(\rho_c) - q^n(0)\big) & \text{otherwise,} \end{cases} \tag{39}$$

The point[a]

[a] Colombo and Goatin 2007.

Υ *never increases* at an interaction, except at a q-jump, where it rises by at most $5\,|\Delta q|$
$\Rightarrow$ uniform BV bound $\Rightarrow$ compactness

[21] weights 4 and 5 are derived not tuned, notice that $4(f_c - q) = 2\,|\Psi(\hat\rho) - \Psi(\check\rho)|$
[22] with $\rho^n(0^-) > \rho_c > \rho^n(0^+)$ and $f(\rho^n(0^+)) = f(\rho^n(0^-)) = q^n(0)$

Well-posedness

Existence and uniqueness theorem[a]

[a] Colombo and Goatin 2007.

For $\rho_0 \in \mathrm{BV}(\mathbb{R})$, $q \in \mathrm{BV}(\mathbb{R}^+; [0, f_c])$, constrained problem (LWR) with $f(\rho(t,0)) \leq q(t)$ admits a unique constrained entropy solution, i.e.

$$\begin{aligned}&\int_{\mathbb{R}_+}\int_{\mathbb{R}} \left(|\rho - k|\partial_t \phi + \operatorname{sgn}(\rho - k)(f(\rho) - k)\partial_x \phi\right) dxdt \\ &+2\int_{\mathbb{R}_+}\left(1 - \frac{q(t)}{f_c}\right) f(k)\phi(t,0)dt + \int_{\mathbb{R}} |\rho_0 - k|\phi(0,x)dx \geq 0,\end{aligned} \tag{40}$$

for every test function $\varphi \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R})$ with L^1 stability estimate

$$\|\rho(t) - \sigma(t)\|_{L^1} \leq \|\rho_0 - \sigma_0\|_{L^1} + 5\int_0^t \left|\frac{dq}{dt}(s)\right| ds, \quad \forall t \in \mathbb{R}_+, \tag{41}$$

Two routes to existence

- **Direct:** wave-front tracking with constrained solver $\mathcal{R}^q$
- **Regularised:** replace the gate by a thin *slab* of reduced capacity and let $\varepsilon \to 0$

Regularised model

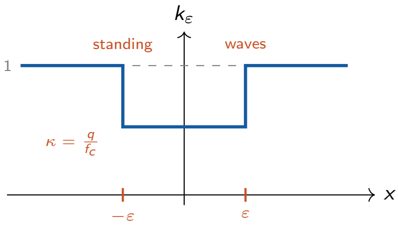
Replace gate by a capacity notch of width 2ε

$$k_\varepsilon(t,x) = \begin{cases} \kappa(t) = \dfrac{q(t)}{f_c}, & |x| < \varepsilon, \\ 1, & |x| > \varepsilon, \end{cases} \tag{42}$$

and solve the discontinuous-flux law

$$\partial_t \rho^\varepsilon + \partial_x\big(k_\varepsilon f(\rho^\varepsilon)\big) = 0, \tag{43}$$

- Two standing waves at $x = \pm\varepsilon$ (speed 0)
- Flux continuity: $k^- f(\rho^-) = k^+ f(\rho^+)$
- Max flux inside the slab is $\kappa f_c = q$



- As $\varepsilon \to 0$, slab collapses onto $x = 0$
- Two standing waves concentrate into a non-classical shock

Two toll-gate model

The problem

$$\partial_t \rho + \partial_x f(\rho) = 0, \quad f(\rho(t,0)) \le q_1(t), \quad f(\rho(t,G)) \le q_2(t), \quad G > 0, \tag{44}$$

with $q_1, q_2 \in \mathrm{BV}(\mathbb{R}^+; [0, f_c])$ two independent capacities

- Gates at $x = 0$ and $x = G$ split the road into three zones
- Two control levers $q_1(t)$, $q_2(t)$: a coordination [23] problem
- Waves generated at one gate travel to the other and interact in $[0, G]$

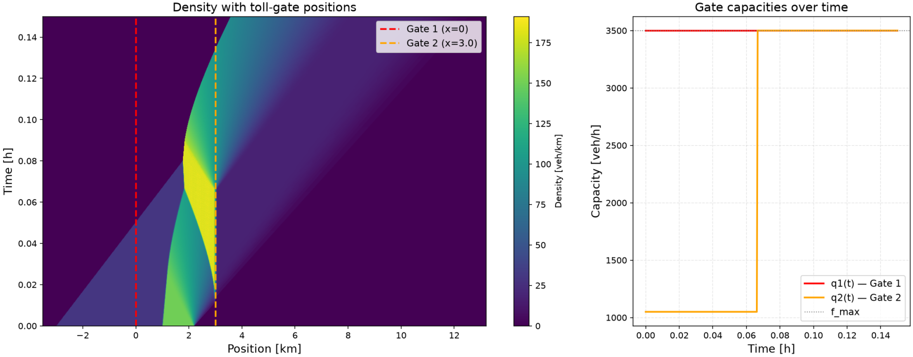
The worry

Since the gates interact [a], are there new interaction cases?

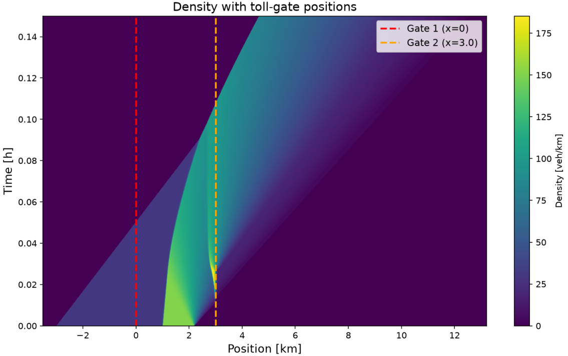
[a] gate 2 sends congestion waves backwards into gate 1 while gate 1 sends waves forward into gate 2

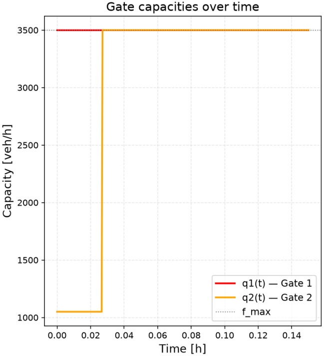
[23] if gate 1 releases too much, gate 2 must absorb it, else $[0, G]$ saturates

First illustration: poor coordination

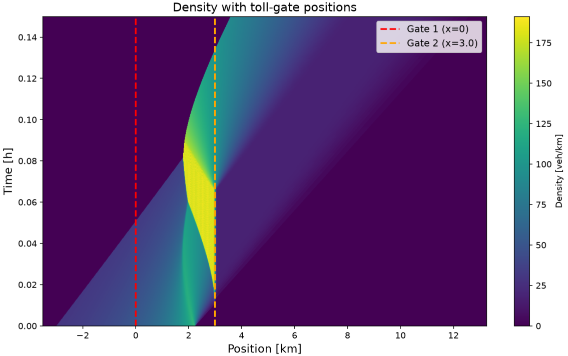


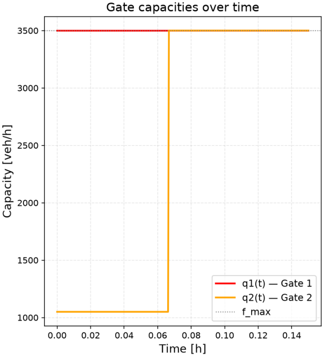
First illustration: some coordination



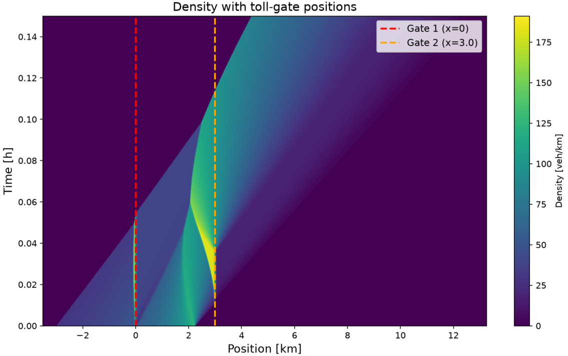


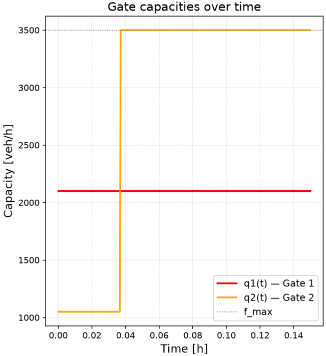
Second illustration: poor coordination





Second illustration: some coordination

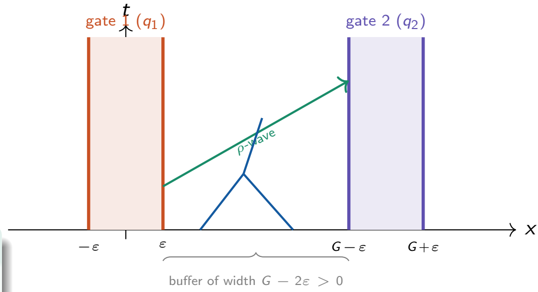




Regularised two-gate model

Two capacity notches

$$k_\varepsilon(t,x) = \begin{cases} \kappa_1(t), & |x| < \varepsilon, \\ \kappa_2(t), & |x - G| < \varepsilon, \\ 1, & \text{else}, \end{cases} \tag{45}$$



Four standing waves, at $\pm\varepsilon$ and $G \pm \varepsilon$.

Assumption (H)

$$\varepsilon < G/2, \tag{46}$$

$\Rightarrow$ the two slabs are disjoint

Approximate initial-value problem

$$\begin{cases} \partial_t \rho^\varepsilon(t,x) + \partial_x \left(k_\varepsilon(t,x)\rho^\varepsilon(t,x)v(\rho^\varepsilon(t,x))\right) = 0, & x \in \mathbb{R}, \quad t \in \mathbb{R}_+, \\ \rho^\varepsilon(0,x) = \rho_0(x), & x \in \mathbb{R}, \end{cases} \tag{47}$$

The answer: dynamic coupling, but functional locality

Two senses of "coupling"

- The gates **are** dynamically coupled:
  $\Rightarrow$ waves really *do travel* between them and the solutionn genuinely links the two
- The gates **are not** functionally coupled:
  $\Rightarrow$ every interaction *estimate is local* to one gate

Why no interaction touches both gates

- The four standing waves never meets: they are *stationary* at *distinct fixed* positions
- An interaction happens at one gate the other gate is outside the finite-speed cone
- A wave travelling from gate 1 to gate 2 is, *on arrival*, just an ordinary incoming wave

Main results

Interaction estimate

Under (H), at *any* interaction

$$\Delta\Upsilon_\varepsilon \;\le\; 5\,|\Delta q_1| + 5\,|\Delta q_2|\,, \tag{48}$$

with $\Delta\Upsilon_\varepsilon \le 0$ away from constraint jumps

Uniform total-variation bound

$$\Upsilon_\varepsilon\big((\rho^\varepsilon)^n(t)\big) \;\le\; \mathrm{TV}(\Psi\circ\rho_0) \;+\; \underbrace{16\,f_c}_{\text{4 standing waves}} \;+\, 5\,\mathrm{TV}(q_1^n) \;+\; 5\,\mathrm{TV}(q_2^n), \tag{49}$$

uniform in the mesh n *and* in ε

$\Rightarrow$ Helly compactness

$\Rightarrow$ existence of the regularised solution ρ^ε, for each $\varepsilon < G/2$

Well-posedness theorem

For $G > 0$, $\rho_0 \in \mathrm{BV}(\mathbb{R})$ and $q_1, q_2 \in \mathrm{BV}(\mathbb{R}^+; [0, f_c])$ then

- **Existence** of a constrained entropy solution [a]

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}} (|\rho - k| \partial_t \phi + \mathrm{sgn}(\rho - k)(f(\rho) - k)\partial_x \phi)\, dxdt$$
$$+2 \int_{\mathbb{R}_+} \left(1 - \frac{q_1(t)}{f_c}\right) f(k)\phi(t, 0)dt + 2\int_{\mathbb{R}_+} \left(1 - \frac{q_2(t)}{f_c}\right) f(k)\phi(t, G)dt \qquad (50)$$
$$+ \int_{\mathbb{R}} |\rho_0 - k|\phi(0, x)dx \geq 0,$$

  for every test function $\varphi \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R})$

- **Uniqueness and L^1-stability** [b]

$$\|\rho(t) - \sigma(t)\|_{L^1} \leq \|\rho_0 - \sigma_0\|_{L^1} + 5\int_0^t \left(\left|\frac{dq_1}{dt}(s)\right| + \left|\frac{dq_2}{dt}(s)\right| \right) ds, \qquad (51)$$

[a] exists directly or as $\lim_{\varepsilon \to 0} \rho^\varepsilon$

[b] doubling of variables, cut-off vanishing near *both* 0 and G and cross terms cancel since $\varepsilon < G/2$

Main results

Convergence theorem

For $\rho_0 \in \mathrm{BV}(\mathbb{R})$, $q_1, q_2 \in \mathrm{BV}(\mathbb{R}^+; [0, f_c])$. For $\varepsilon < G/2$, ρ^ε solve regularised problem, then

$$\rho^\varepsilon \xrightarrow[\varepsilon \to 0]{L^1_{\mathrm{loc}}} \rho, \tag{52}$$

where ρ is the unique constrained entropy solution of the two-gate problem

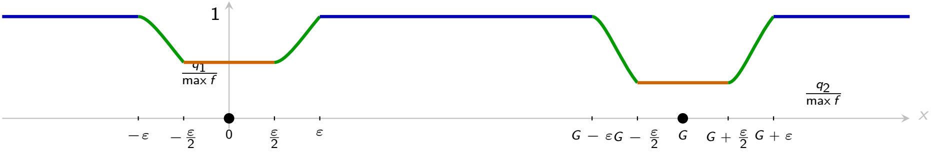
- k^ε has jump discontinuities in both t (through q_k) and x (at $\pm\varepsilon$ and $G \pm \varepsilon$)
- Introduce regularised gate function

$$\phi^\varepsilon(t,x) \coloneqq 1 - \left(1 - \frac{q_1(t)}{\max f}\right)\theta_\varepsilon(x) - \left(1 - \frac{q_2(t)}{\max f}\right)\theta_\varepsilon(x-G), \quad x \in \mathbb{R}, \quad t \in \mathbb{R}^+, \tag{53}$$

where θ_ε is a suitable bump function

Second approximation problem

(a) $k^{\varepsilon}(t,x)$.



(b) $\phi^{\varepsilon}(t,x)$.

Top: Representation of $k^{\varepsilon}(t,x)$. Bottom: Representation of $\phi^{\varepsilon}(t,x)$.

Approximate initial-value problem

$$\begin{cases} \partial_t \rho^{\varepsilon}(t,x) + \partial_x \left(\phi^{\varepsilon}(t,x)\rho^{\varepsilon}(t,x)v(\rho^{\varepsilon}(t,x))\right) = 0, & x \in \mathbb{R}, \quad t \in \mathbb{R}_+, \\ \rho^{\varepsilon}(0,x) = \rho_0(x), & x \in \mathbb{R}, \end{cases} \tag{54}$$

Corresponding microscopic model

- Consider microscopic model (ODE system)

$$
\begin{cases}
\frac{dx_N^{\varepsilon,N}}{dt}(t) = v_{\max}\phi^\varepsilon(t, x_N^{\varepsilon,N}(t)), & t \in \mathbb{R}, \\
\frac{dx_i^{\varepsilon,N}}{dt}(t) = v\left(\frac{L}{N\left(x_{i+1}^{\varepsilon,N}(t) - x_i^{\varepsilon,N}(t)\right)}\right)\phi^\varepsilon(t, x_i^{\varepsilon,N}(t)), & t \in \mathbb{R}, \quad i = 0, \cdots, N-1, \\
x_i^{\varepsilon,N}(0) = \bar{x}_i, & i = 0, \cdots, N,
\end{cases}
\tag{55}
$$

where $\bar{x}_i$ denotes initial position of vehicle i, ϕ^ε is given in (53) and

$$
L := \|\rho_0\|_{L^1(\mathbb{R})}, \tag{56}
$$

- Velocity v satisfies
  1. $v \in C^1([0, +\infty))$,
  2. v is decreasing on $[0, +\infty)$,
  3. $v(0) = v_{\max}$ for some $v_{\max} \in \mathbb{R}$,
  4. $\rho \mapsto \rho v'(\rho)$ is non-increasing from $[0, +\infty)$ to $(-\infty, 0]$,

The catch

Control theory for traffic assumes the density $\rho(t, \cdot)$ is known at all time t
$\Rightarrow$ in practice it is never measured directly

Reconstruction of traffic flow

$n \ll N$ — only initial and final positions are collected

n **probe vehicles observed** (GPS) among N **total**

The plan

- Reconstruct ρ from sparse probe data
- Use reconstructed density to control flow between the gates
- Obtain theoretical guarantees[a] (mainly compactness and convergence)

[a] Baloul, Hayat, Liard, and Lissy 2025.

ODE-constrained optimisation

- Consider parametrised microscopic model [24] with finite time horizon T

$$\begin{cases} \frac{dx_n^{\varepsilon,N}}{dt}(t) = v_{\max}\phi^{\varepsilon}(t, x_n^{\varepsilon,N}(t)), & t \in (0,T], \\ \frac{dx_i^{\varepsilon,N}}{dt}(t) = v\left(\frac{\alpha_i^N L}{N\left(x_{i+1}^{\varepsilon,N}(t) - x_i^{\varepsilon,N}(t)\right)}\right)\phi^{\varepsilon}(t, x_i^N(t)), & t \in (0,T], \quad i = 0,\cdots,n-1, \\ x_i^{\varepsilon,N}(0) = \bar{x}_i, & i = 0,\cdots,n, \end{cases} \tag{57}$$

- Introduce feasible set (encoding physical [25] conditions)

$$\mathcal{A}^N := \left\{\alpha^N \in \mathbb{R}^n : \quad \alpha_i^N \in [1, \bar{z}_i], \quad \sum_{j=0}^{n-1} \alpha_j^N = N\right\}, \tag{58}$$

- ODE constrained optimisation

$$\underset{\alpha^N \in \mathcal{A}^N.}{\text{minimize}} \quad \frac{1}{2}\sum_{i=0}^{n} |x_i(T) - \bar{y}_i|^2 \quad \text{s.t.} \quad (57) \text{ holds} \tag{59}$$

---

[24] initial $\bar{x}_i$ and final $\bar{y}_i$ probe positions

[25] $\bar{z}_i := \min\{N(\bar{x}_{i+1} - \bar{x}_i)/L, N(\bar{y}_{i+1} - \bar{y}_i)/L\}$

Main result

Define discrete local (Euler) density as

$$\rho_i^{\varepsilon,N}(t) := \frac{\left(\alpha_i^N\right)^* L}{N\left(x_{i+1}^{\varepsilon,N}(t) - x_i^{\varepsilon,N}(t)\right)}, \quad t \in (0,T], \quad i = 0, \cdots, n-1, \tag{60}$$

Many-particle limit theorem

Let (v1)–(v4) hold, $\rho_0 \in \mathcal{M}_L{}^a \cap L^\infty$ and assume

$$\max_i\{\alpha_i\} = o(N), \quad (N \to \infty), \tag{61}$$

Fix $\varepsilon > 0$. Then, for $N \geq N_0(\varepsilon)$,

$$\widehat{\rho}^{\,\varepsilon,N}(t,x) = \sum_{i=0}^{n-1} \rho_i^\varepsilon(t)\,\chi_{[x_i(t),x_{i+1}(t))}(x) \xrightarrow[N\to\infty]{} \rho^\varepsilon \quad \text{in } L^1_{\mathrm{loc}}, \tag{62}$$

where ρ^ε is the unique entropy solution of $\partial_t \rho^\varepsilon + \partial_x\left(\phi^\varepsilon \rho^\varepsilon v(\rho^\varepsilon)\right) = 0$

[a] $\mathcal{M}_L := \{\mu$ Radon measure on $\mathbb{R}$ with compact support and $\mu \geq 0, \quad \mu(\mathbb{R}) = L\}$

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- Segment $[0, G]$, $G > 0$, with a toll gate at each end governed by

$$\begin{aligned} \partial_t \rho + \partial_x f(\rho) &= 0, \quad x \in (0, G), \quad t \in \mathbb{R}_+, \\ f(\rho(t, 0)) &\leq F_1(t, \rho), \quad t \in \mathbb{R}_+, \\ f(\rho(t, G)) &\leq F_2(t, \rho), \quad t \in \mathbb{R}_+, \\ \rho(0, x) &= \rho_0(x), \quad x \in (0, G), \end{aligned} \tag{63}$$

- **New feature**: F_1, F_2 are no longer fixed constants [26]
  $\Rightarrow$ they are feedback controllers, allowed to depend on the current state $\rho(t, \cdot)$ itself
- **Task**: choose F_1, F_2 so that $\rho(t, \cdot)$ is driven (and stays) at prescribed target profile w

[26] or solely time-dependent functions

The control target

- Fix $x_0 \in (0, G)$ and $\rho_l, \rho_r \in (0, \rho_{\max})$ with $\rho_l < \rho_r$ and $f(\rho_l) = f(\rho_r) < f(\rho_c)$
- Target profile [27]

$$w(x) = \begin{cases} \rho_l, & 0 \leq x < x_0, \\ \rho_r, & x_0 < x \leq G, \end{cases} \tag{64}$$

- **Objective**: for *any* initial datum ρ_0, design F_1, F_2 such that the $\mathcal{G}$-solution satisfies

$$\rho(t, \cdot) \xrightarrow[t \to \infty]{L^1(0,G)} w, \tag{65}$$

[27] a stationary shock sitting *inside* the segment at x_0

Admissibility germ

- Constraint [28] $f(\rho(t,0)) \leq F_1$ forces a shock that violates the Oleinik condition
- Existence of $\mathcal{G}$-entropy solutions
  $\Rightarrow$ Kruzhkov entropy away from $\{0, G\}$ plus an admissibility germ condition

Definition (Admissibility germ[a])

[a] Andreianov, Goatin, and Seguin 2010.

For $F \in [0, f(\rho_c)]$ the admissibility germ for the constrained conservation law is

$$\mathcal{G}(F) = \mathcal{G}_1(F) \cup \mathcal{G}_2(F) \cup \mathcal{G}_3(F) \subset [0, \rho_{\max}] \times [0, \rho_{\max}], \tag{66}$$

of trace pairs (c_l, c_r), where

$$\begin{aligned}
\mathcal{G}_1(F) &= \big\{(c_l, c_r) : c_l > c_r,\ f(c_l) = f(c_r) = F\big\},\\
\mathcal{G}_2(F) &= \big\{(c, c) : f(c) \leq F\big\},\\
\mathcal{G}_3(F) &= \big\{(c_l, c_r) : c_l < c_r,\ f(c_l) = f(c_r) \leq F\big\}.
\end{aligned}$$

- $\mathcal{G}(F)$ encodes exactly which traces (ρ_l, ρ_r) are admissible for a given flux bound F
- Machinery is applied twice:
  once at $x = 0$ with bound $F_1(t, \rho)$, once at $x = G$ with bound $F_2(t, \rho)$

[28] (resp. $f(\rho(t, G)) \leq F_2$)

Designing the feedback

- Track the (signed) L^1 distance to the target

$$m(t) := \int_0^G \big(\rho(t,x) - w(x)\big)\, dx, \tag{67}$$

- Drive m to 0 through an auxiliary ODE

$$\frac{dm}{dt}(t) = -g(m(t)), \quad m(0) = m_0, \tag{68}$$

with g continuous, $g(0) = 0$ and $x\,g(x) > 0$ for $x \neq 0$

- **Feedback controllers**[29]

$$F_1(t,\rho) = \max\big(\min(f(\rho_l) - g(m(t)),\, f(\rho_c) - \varepsilon),\, 0\big), \quad F_2(t,\rho) = f(\rho_r), \tag{69}$$

- **Interpretation**: the exit gate is held exactly at the target flux
the entry gate is the one doing the active correction adjusted by current error $m(t)$

[29] Liard, Marx, and Perrollaz 2023.

Main results

Existence theorem[a]

[a] Liard, Marx, and Perrollaz 2023.

For any $\rho_0 \in L^\infty(\mathbb{R}; [0, \rho_{\max}])$, there exists a $\mathcal{G}$-entropy solution ρ to control problem

Asymptotic stability theorem[a]

[a] Liard, Marx, and Perrollaz 2023.

For any $\rho_0 \in L^\infty([0, G]; [0, \rho_{\max}])$, ρ converges [a] to w in $L^1(0, G)$ as $t \to \infty$

[a] convergence to an *arbitrary* interior stationary shock for *any* initial condition

- Two refinements depending on g
  - $g(x) = K_0 x$ with $K_0 > 0 \Rightarrow$ exponential convergence, i.e.

$$\exists C > 0, \quad \|\rho(t, \cdot) - w\|_{L^1} \leq Ce^{-K_0 t}\|\rho_0 - w\|_{L^1} + C, \tag{70}$$

  - g with $\int_0^\alpha \frac{dy}{g(y)} > -\infty$ with $\alpha \in \mathbb{R}^* \Rightarrow$ finite-time convergence, i.e.

$$\exists \bar{T}, \quad \rho(t, \cdot) = w, \quad t \geq \bar{T}, \tag{71}$$

Open question

- Feedback law needs $m(t) = \int_0^G (\rho(t,x) - w(x))\, dx$ at every instant
  $\Rightarrow$ requires the *full* density profile
- In practice, only a handful of sensors: situation from reconstruction part
- **Perspective**: replace $\rho(t,\cdot)$ in $m(t)$ by the reconstructed density $\rho^{\varepsilon,N}(t,\cdot)$
  $\Rightarrow$ assess[30] whether the same convergence to w still holds

[30] ongoing work with Agbo Bidi, K.

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Conclusion

- Traffic congestion (though chaotic) is governed by a single conservation principle
- A single local constraint is already enough to break classical entropy theory
- Non-classical shocks appear and a genuinely new $\mathcal{G}$-admissibility notion
- Under well-posed model, limited measurements is enough to reconstruct the state
- Two toll gates are enough to stabilize towards interior stationnary shock target
- Simultaneously a modeling singularity, an observation point and a control lever

Perspectives

- Conservation law with moving bottleneck[31] (slow vehicle)

$$\begin{cases} \partial_t \rho(t,x) + \partial_x f(\rho)(t,x) = 0, & x \in \mathbb{R}, \quad t \in \mathbb{R}^+, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}, \\ f\left(\rho(t,y(t))\right) - \frac{dy}{dt}(t)\rho(t,y(t)) \le \frac{\alpha \rho_{\max}}{4 v_{\max}} \left(v_{\max} - \frac{dy}{dt}(t)\right)^2, & t \in \mathbb{R}^+, \\ \frac{dy}{dt}(t) = \omega\left(\rho(t, y(t)_+)\right), & t \in \mathbb{R}^+, \\ y(0) = y_0, & \end{cases} \tag{72}$$

- Network with a junction[32] J and N incoming roads and M outgoing ones

$$\begin{cases} \partial_t \rho_l(t,x) + \partial_x \left(f(\rho_l(t,x))\right) = 0, & t \in \mathbb{R}^+, \quad x \in I_l, \quad l = 1, \dots, N+M, \\ \rho_l(0,x) = \rho_{0,l}(x), & x \in I_l = [a_l, b_l], \quad l = 1, \dots, N+M, \end{cases} \tag{73}$$

$\Rightarrow \sum_{i=1}^{N} f\left(\rho_i(t,(b_i)_-)\right) = \sum_{j=N+1}^{N+M} f\left(\rho_j(t,(a_j)_+)\right)$ (Rankine Hugoniot)
$\Rightarrow \sum_{i=1}^{N} f\left(\rho_i(t,(b_i)_-)\right)$ is maximised with $f\left(\rho_j(\cdot,(a_j)_+)\right) = \sum_{i=1}^{N} a_{j,i} f\left(\rho_i(\cdot,(b_i)_-)\right)$
$\Rightarrow$ existence and uniqueness theory requires notion of Riemann germ[33]

- Extension to second-order models[34] (Payne-Whitham, GARZ)

[31] Liard and Piccoli 2021.
[32] Coclite, Piccoli, and Garavello 2005.
[33] Monneau 2024.
[34] Hayat, Liard, Marcellini, and Piccoli 2023.

Traffic jams are a costly, emergent phenomenon[35]

Table 16: Annual hours wasted (excl. PTI) and fuel consumed per vehicle whilst idling, 2013

Country	Annual hours wasted (2013)	Fuel consumed in litres per vehicle (2013)
UK	40.1	72.06
France	43.9	78.95
Germany	38.2	68.62
US	22.0	39.53
City	**Annual hours wasted (2013)**	**Fuel consumed in litres per vehicle (2013)**
London	81.6	146.59
Paris	55.1	99.01
Stuttgart	60.2	108.19
Los Angeles	64.3	115.58

Source: INRIX, Cebr analysis

Table 18: CO_2 equivalent emissions from fuel wasted as a result of congestion, kilotons of CO_2 equivalent

Country	2013	2030	2013-30 % change
UK	1,931	2,401	24
France	1,917	2,175	13
Germany	3,010	3,032	1
US	8,576	10,351	21
Advanced Economies	15,434	17,959	16
City	**2013**	**2030**	**2013-30 % change**
London	658	937	42
Paris	773	904	17
Stuttgart	253	260	3
Los Angeles	1,433	1,838	28
Cities	3,117	3,939	26

Source: INRIX, Cebr analysis

Table 14: City-level: economy-wide direct and indirect costs to households, millions of Dollars

City	Sector	2013	2020	2025	2030	2013-30 % change
London	Direct costs (value of fuel & time wasted)	4,310	5,602	6,669	7,741	80
	Indirect costs (increased cost of doing business)	4,203	5,088	5,943	6,779	61
	Total	**8,513**	**10,690**	**12,612**	**14,520**	**71**
Paris	Direct costs (value of fuel & time wasted)	6,282	7,558	8,688	10,008	59
	Indirect costs (increased cost of doing business)	5,410	6,404	7,479	8,693	61
	Total	**11,692**	**13,963**	**16,167**	**18,701**	**60**
Stuttgart	Direct costs (value of fuel & time wasted)	2,054	2,287	2,496	2,694	31
	Indirect costs (increased cost of doing business)	1,116	1,242	1,396	1,549	39
	Total	**3,170**	**3,529**	**3,892**	**4,242**	**34**
LA	Direct costs (value of fuel & time wasted)	13,213	17,305	20,074	22,294	69
	Indirect costs (increased cost of doing business)	9,987	12,610	14,573	16,061	61
	Total	**23,200**	**29,915**	**34,647**	**38,355**	**65**

Source: Cebr analysis

Table 19: CO_2 equivalent emissions in monetary terms, millions of Dollars[25]

Country	2013	2030
UK	10.5	286.3
France	13.9	308.4
Germany	21.8	429.9
US	300.2	538.2
City	**2013**	**2030**
London	3.6	111.7
Paris	5.6	128.2
Stuttgart	1.8	36.9
Los Angeles	50.2	95.6

Source: INRIX, Cebr analysis

[35] INRIX, Economics, and Research 2014.

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Entropy conditions for scalar conservation laws

Consider the scalar conservation law

$$\partial_t \rho + \partial_x f(\rho) = 0, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}^+, \tag{74}$$

Lax–Oleĭnik condition: primarily 1D, requires convexity or concavity of f

- Oleĭnik E-condition

$$\rho(x+z,t) - \rho(x,t) \leq \frac{C}{t} z, \quad z > 0, \tag{75}$$

- Lax entropy condition: for a shock $\rho_L \to \rho_R$ with speed s

$$f'(\rho_L) > s > f'(\rho_R), \tag{76}$$

Kružkov entropy condition: in any dimension, no convexity needed

- Kružkov entropy solution: for all k,

$$\partial_t |\rho - k| + \partial_x \big(\operatorname{sgn}(\rho - k)(f(\rho) - f(k))\big) \leq 0, \tag{77}$$

  in the sense of distributions
- Gives existence, uniqueness, stability

Relation: In 1D with convex (or concave) flux, both conditions are equivalent and select the same unique entropy solution but Kružkov's framework is the general one

Numerical approximation: the Godunov scheme

- **Idea**: at each interface solve the exact Riemann problem between the two neighboring cell values, then average
- Discretize space into cells x_i, time into steps t^n, with cell averages $\rho_i^n \approx \rho(t^n, x_i)$
- **Update formula** (finite volume, conservative form)

$$\rho_i^{n+1} = \rho_i^n - \frac{\Delta t}{\Delta x}\Big(F(\rho_i^n, \rho_{i+1}^n) - F(\rho_{i-1}^n, \rho_i^n)\Big) \tag{78}$$

- **Godunov numerical flux**: exact Riemann solver value at the interface

$$F(\rho_l, \rho_r) = \begin{cases} \min\limits_{\rho\in[\rho_l,\rho_r]} f(\rho), & \rho_l \leq \rho_r, \\ \max\limits_{\rho\in[\rho_r,\rho_l]} f(\rho), & \rho_l > \rho_r, \end{cases} \tag{79}$$

  For a concave f: reduces to $\min(f(\rho_l), f(\rho_r))$ or $f(\bar{\rho})$ when $\bar{\rho} \in [\rho_r, \rho_l]$
- **CFL condition** for stability:

$$\Delta t \leq \frac{\Delta x}{\max_i |f'(\rho_i^n)|}, \tag{80}$$

- Conservative by construction and converges to the entropy solution as $\Delta x, \Delta t \to 0$